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Phase structure of matrix quantum mechanics at finite temperature

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Abstract: We study matrix quantum mechanics at finite temperature by Monte Carlo simulation. The model is obtained by dimensionally reducing 10d $\mathrm{U}(N)$ pure Yang-Mills theory to 1d. Following Aharony et al., one can view the same model as describing the high temperature regime of (1+1)d $\mathrm{U}(N)$ *super* Yang-Mills theory on a circle. In this interpretation an analog of the deconfinement transition was conjectured to be a continuation of the black-hole/black-string transition in the dual gravity theory. Our detailed analysis in the critical regime up to $N = 32$ suggests the existence of the non-uniform phase, in which the eigenvalue distribution of the holonomy matrix is non-uniform but gapless. The transition to the gapped phase is of second order. The internal energy is constant (giving the ground state energy) in the uniform phase, and rises quadratically in the non-uniform phase, which implies that the transition between these two phases is of third order.

Keywords: Nonperturbative Effects, M(atrix) Theories, Gauge-gravity correspondence.

Contents

1. Introduction

Recently large-N gauge theories are playing increasingly important roles in string theory. One of the crucial discoveries was that $\mathrm{U}(N)$ gauge theory appears as a low energy effective theory [1] for a stack of N D-branes [2] in string theory. This led to various interesting conjectures concerning large-N gauge theories. For instance, it is conjectured that large-N gauge theories obtained by dimensionally reducing 10d $\mathrm{U}(N)$ super Yang-Mills theory to $D = 0, 1, 2$ dimensions provide non-perturbative formulations of superstring/M theories. These are called Matrix theory ($D = 1$) [3], the IIB matrix model ($D = 0$) [4] and matrix string theory ($D = 2$) [5], respectively. Another type of conjectures asserts the duality between strongly coupled large-N gauge theory and weakly coupled supergravity. In the AdS/CFT correspondence [6–8], for instance, it is conjectured that 4-dimensional $\mathrm{U}(N)$ $\mathcal{N} = 4$ super Yang-Mills theory is dual to the type IIB supergravity on $\mathrm{AdS}_5 \times \mathrm{S}^5$. Generalizing this correspondence to the finite temperature case, it was argued that the deconfinement phase transition on the gauge theory side corresponds to the Hawking-Page transition on the gravity side [9]. This has been further extended to non-conformal gauge theories including supersymmetric matrix quantum mechanics [10]. See refs. [11–20] for extensive studies on the relationship between large-N gauge theories at finite temperature and the black-hole physics.

In this paper we study matrix quantum mechanics at finite temperature by Monte Carlo simulation. The model is obtained formally by dimensionally reducing 10d $\mathrm{U}(N)$

pure Yang-Mills theory to 1d, and it may be viewed[1] as the bosonic part of the low energy effective theory of D-particles in type IIA superstring theory [1]. By considering the Euclidean time direction as a spatial direction instead, one can view the bosonic model as describing the high temperature regime of (1+1)d $\mathrm{U}(N)$ $\mathcal{N}=8$ *super* Yang-Mills theory on a circle. In this interpretation an analog of the deconfinement transition was speculated [15] to be a continuation of the black-hole/black-string phase transition in the dual gravity theory.[2]

Unlike previous works, our detailed analysis in the critical regime up to $N=32$ suggests the existence of the non-uniform phase, in which the eigenvalue distribution of the holonomy matrix is non-uniform but gapless. The transition to the gapped phase appears to be of second order. At low temperature, the internal energy is constant (giving the ground state energy) as a result of the Eguchi-Kawai equivalence. We use this property to identify the uniform phase. As one enters the non-uniform phase increasing the temperature, the internal energy starts to rise quadratically. This implies that the transition between the uniform phase and the non-uniform phase is of third order. These observations select one of the two scenarios suggested by the Landau-Ginzburg analysis [15].

The rest of this paper is organized as follows. In section 2 we define the model and the observables we study. In section 3 we present an overview of the phase diagram as seen from explicit results for the observables. In section 4 we focus on the critical regime and study the phase structure in more detail. In section 5 we use our results to speculate on the phase diagram of (1+1)d super Yang-Mills theory on a circle. Section 6 is devoted to a summary and discussions. In appendix A we derive a formula for the internal energy, which is used for its evaluation. In appendix B we present the details of our simulation.

2. The model and observables

The model we study in this paper is defined by the action

$$S=\frac{1}{g^2}\int_0^\beta dt\,\mathrm{tr}\left\{\frac{1}{2}\Big(D_t X_i(t)\Big)^2-\frac{1}{4}[X_i(t),X_j(t)]^2\right\}, \tag{2.1}$$

where D_t represents the covariant derivative $D_t\equiv\partial_t-i\,[A(t),\ \cdot\ ]$. The one-dimensional fields $A(t)$ and $X_i(t)$ ($i=1,2,\ldots,9$) are $N\times N$ Hermitian matrices, which can be regarded as the gauge field and nine adjoint scalars, respectively. The model can be formally obtained by dimensionally reducing 10d $\mathrm{U}(N)$ pure Yang-Mills theory to 1 dimension. The Euclidean time t has a finite extent β, which corresponds to the inverse temperature $\beta=1/T$, and all the fields obey periodic boundary conditions.

[1]One may also consider the model as the "bosonic Matrix theory" [21], but in that case, the compactified Euclidean time direction actually corresponds to a light-cone coordinate, which might make the meaning of "finite temperature" a bit subtle unlike in the other interpretations. See refs. [22], however.

[2]Some of the results in refs. [15] were already anticipated in a pioneering work [14] on the phase structure of black holes on a circle. There, the relationship of the supergravity solutions to non- and near-extremal branes on a circle and to the corresponding dual non-gravitational theories are discussed.

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The 't Hooft coupling constant $\lambda \equiv g^2 N$ has the dimension of $(\text{mass})^3$, and we fix it when we take the large-N limit.[3] The physical properties of the system depend only on the dimensionless effective coupling constant given by[4]

$$\lambda_{\text{eff}} \equiv \frac{\lambda}{T^3} \, . \tag{2.2}$$

In what follows we set $\lambda = 1$ without loss of generality.

It is known that the bosonic matrix quantum mechanics undergoes a phase transition [21, 15, 23] at some critical temperature, which can be interpreted as the Hagedorn transition in string theory [24, 16]. This transition is associated with the spontaneous breakdown of the $\mathrm{U}(1)$ symmetry

$$A(t) \mapsto A(t) + \alpha \mathbf{1} \, , \tag{2.3}$$

and therefore it is analogous to the deconfinement transition in ordinary gauge theories. The order parameter is given by the Polyakov line

$$P \equiv \frac{1}{N} \mathrm{tr}\, U \, , \tag{2.4}$$

$$U \equiv \mathcal{P} \exp \left(i \int_0^\beta dt A(t) \right) \, , \tag{2.5}$$

where the symbol $\mathcal{P} \exp$ represents the path-ordered exponential and the unitary matrix U is called the holonomy matrix. In section 4 we present numerical results suggesting that the "deconfined phase" is further divided into two phases.

As a fundamental quantity in thermodynamics, the free energy $\mathcal{F} \equiv -\frac{1}{\beta} \ln Z(\beta)$ is defined in terms of the partition function

$$Z(\beta) = \int [\mathcal{D}X]_\beta [\mathcal{D}A]_\beta \, \mathrm{e}^{-S(\beta)} \, , \tag{2.6}$$

where the suffix of the measure $[\, \cdot \,]_\beta$ represents the period of the field to be path-integrated. However, the free energy $\mathcal{F}$ cannot be calculated straightforwardly by Monte Carlo simulation because that would require evaluation of the partition function $Z(\beta)$. We therefore study the internal energy defined by

$$E \equiv \frac{d}{d\beta}(\beta \mathcal{F}) = -\frac{d}{d\beta} \log Z(\beta) \, , \tag{2.7}$$

which has equivalent information as the free energy (given that $\mathcal{F} = E$ at $T = 0$). Note also that the internal energy at $T = 0$ provides the ground state energy of the quantum mechanical system. In appendix A we derive a formula

$$\frac{1}{N^2} E = \frac{3}{4} \langle F^2 \rangle \, , \tag{2.8}$$

[3] This is different from the decompactifying limit [3] in the Matrix theory.

[4] One can confirm this statement by rescaling the fields and the coordinate t appropriately so that all the λ and T dependence appears in the combination of eq. (2.2).

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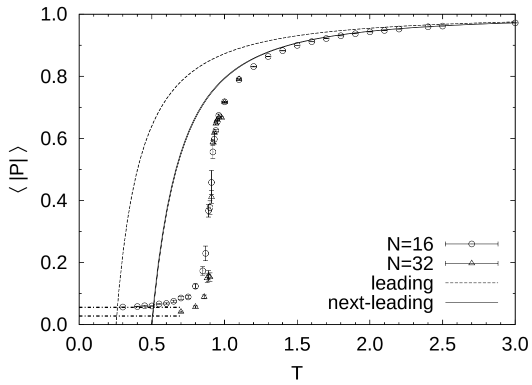

Figure 1: The Polyakov line $\langle |P| \rangle$ is plotted against T for $N = 16$, and in the critical regime, also for $N = 32$. The dashed line and the solid line represent the results of the high temperature expansion [26] for $N = 16$, which are obtained at the leading order and at the next-leading order, respectively.

$$F^2 \equiv -\frac{1}{N\beta} \int_0^\beta dt \, \mathrm{tr} \left([X_i(t), X_j(t)]^2 \right) , \tag{2.9}$$

where the symbol $\langle \cdot \rangle$ represents the expectation value with respect to $Z(\beta)$. This formula enables us to calculate the internal energy E directly by Monte Carlo simulation. As yet another quantity, we study

$$R^2 \equiv \frac{1}{N\beta} \int_0^\beta dt \, \mathrm{tr} \, (X_i)^2 , \tag{2.10}$$

which represents the "extent of space".

The details of our simulation are given in appendix B. The number of sites N_t in the Euclidean time direction is taken to be the smallest integer that satisfies $N_t \geq \frac{1}{0.02T}$ for figures 1 and 2 , and $N_t \geq \frac{1}{0.05T}$ for figures 3, 4 and 5. This corresponds to taking the lattice spacing to be $a \simeq 0.02$ and $a \simeq 0.05$, respectively. See ref. [25] for a systematic study of the finite lattice-spacing effects in an analogous model. Note that the previous work [15] used $N_t = 5$ at any temperature.

3. An overview of the phase diagram

In this section we present an overview of the phase diagram of the model (2.1). Figure 1 shows the results for the Polyakov line. It changes drastically at $T \simeq 0.9$. For $T \gtrsim 0.9$, the results for $N = 16$ and $N = 32$ lie on top of each other, and approaches the maximum value 1 as T increases. The results at $T \gtrsim 2$ are nicely reproduced by the high temperature expansion[5] including the next-leading order terms [26]. For $T \lesssim 0.9$, the expectation value $\langle |P| \rangle$ takes small values, which seem to vanish in the large-N limit as $1/N$.

[5]The results of the high temperature expansion [26] are plotted with the double lines showing the statistical errors coming from the Monte Carlo integration over the zero modes. In some cases, the errors are so small that one cannot recognize the lines as two separate ones.

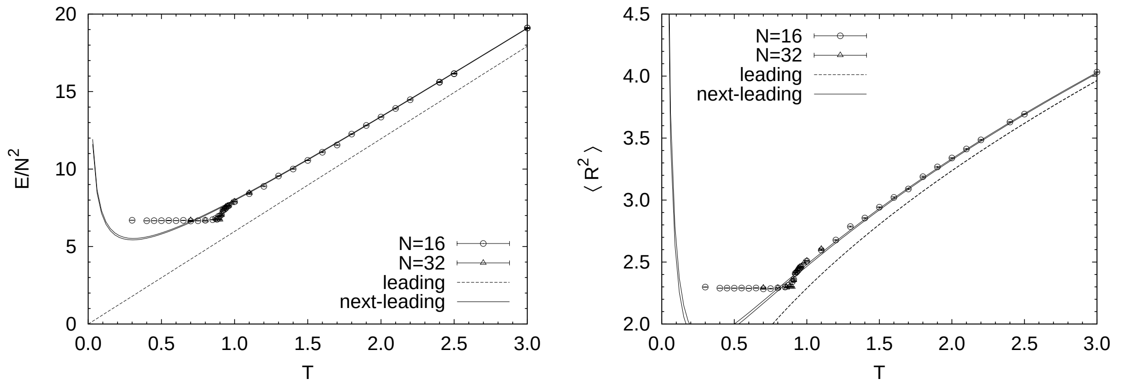

Figure 2: The same as figure 1 but for the internal energy $\frac{1}{N^2}E$ (left) and for the "extent of space" $\langle R^2 \rangle$ (right).

In fact, the expectation value $\langle |P| \rangle$ in the $T \to 0$ limit seems to agree well with the finite-N results obtained by generating the holonomy matrix U randomly using the Haar measure, which are represented by the horizontal dash-dotted lines in figure 1 for $N = 16, 32$. The latter results are seen to be proportional to $1/N$ for $N = 8, 16, 32, 64$. In this way, we can understand the observed $1/N$ behavior of $\langle |P| \rangle$ in the original model at low temperature. On the other hand, finite N effects at $T \gtrsim 0.9$ seems to be much smaller. This can be understood from the fact that in this regime the phenomenological description of the holonomy matrix is given by the Gross-Witten model (4.3), which has finite N effects of the order of $\mathrm{O}(1/N^2)$ for $N \gg \sqrt{\kappa}$.

In figure 2 we plot the internal energy obtained from the formula (2.8), and the "extent of space" $\langle R^2 \rangle$. For $T \lesssim 0.9$, we observe that the results are independent of T. This can be understood as a consequence of the Eguchi-Kawai equivalence. In general it states that the expectation value of a single-trace operator in D-dimensional $\mathrm{U}(N)$ gauge theory is independent of the volume in the large-N limit, provided that the $\mathrm{U}(1)^D$ symmetry is not spontaneously broken [27].[6] The T-independence of $\lim_{N\to\infty} \frac{1}{N^2}E$ at low temperature can also be viewed as a consequence of the fact that only $\mathrm{U}(N)$ invariant states appear in the low energy spectrum in a confining theory.

4. Closer look at the critical regime

In this section we focus on the critical regime and study the phase structure in more detail.

In figure 3 we plot the Polyakov line against temperature magnifying the critical regime. For $N = 16$ the Polyakov line changes smoothly with T. However, for $N = 32$ the behavior of the Polyakov line changes drastically at $T \sim T_{\mathrm{c1}} \equiv 0.905(2)$. In figure 4 we plot the internal energy E and the "extent of space" $\langle R^2 \rangle$ in the critical regime. The

[6] In 3d and 4d pure $\mathrm{SU}(N)$ gauge theory on a torus, the SSB occurs at finite volume [28, 29], and the relation to the deconfinement transition at finite temperature [30] is discussed. See also refs. [31 – 33] for related works.

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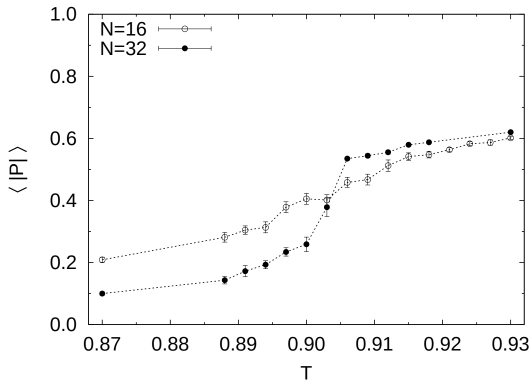

Figure 3: The Polyakov line $\langle|P|\rangle$ is plotted against T in the critical regime for $N = 16$ and $N = 32$. The dotted lines are drawn to guide the eye.

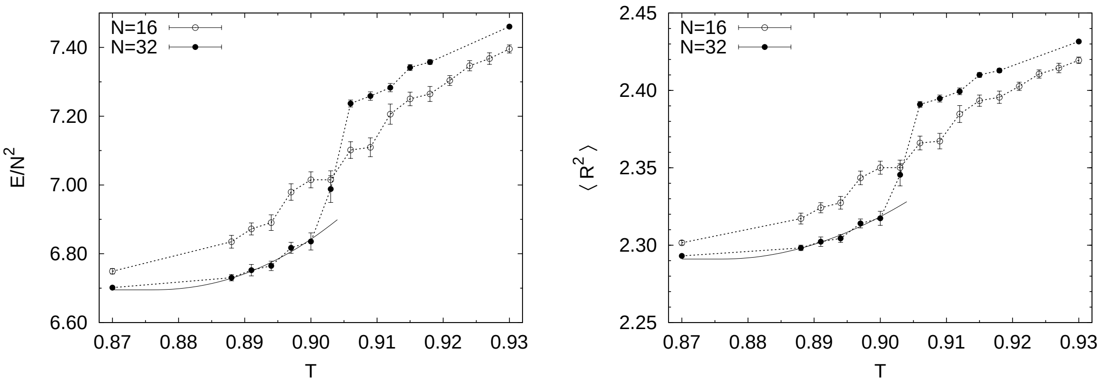

Figure 4: The same as figure 3 but for the internal energy E (left) and for the "extent of space" $\langle R^2\rangle$ (right). The solid line in the left panel is a fit to $\frac{E}{N^2} - \varepsilon_0 = c(T - T_{c2})^p$, where ε_0 is given by the low-temperature data. An analogous fit in the right panel yields consistent values for T_{c2} and p.

qualitative behavior is similar to the Polyakov line. The $N = 32$ data suggest that all the observables depend continuously on T, but their first derivatives with respect to T seem to be discontinuous at $T \sim T_{c1}$. Thus we conclude that there exists a second order phase transition at $T \sim T_{c1}$.

In order to clarify the nature of the phase transition seen above, let us consider the eigenvalues of the holonomy matrix U given by eq. (2.5), which we denote as $e^{i\vartheta_j}$ $(j = 1,\ldots,N)$, where $\vartheta_j \in (-\pi,\pi]$. Then we define the eigenvalue distribution by

$$\rho(\theta) \equiv \frac{1}{N}\sum_{j=1}^{N}\langle\delta\left(\theta-\vartheta_j\right)\rangle\ , \tag{4.1}$$

where we assume that the $\mathrm{U}(1)$ transformation (2.3) is applied to each configuration in

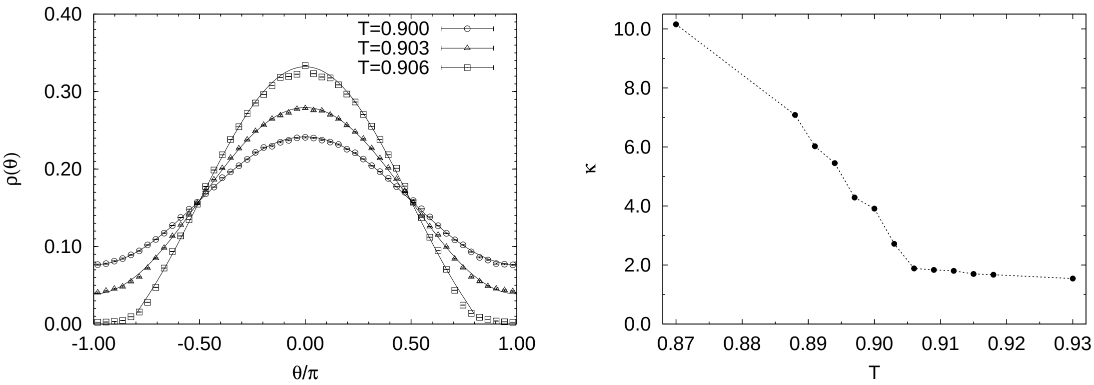

Figure 5: (Left) The eigenvalue distribution $\rho(\theta)$ is plotted for $N = 32$ at three different values of T in the critical regime. The solid lines represent a fit to the Gross-Witten form given by either (4.4) or (4.5) depending on the fitting parameter κ. (Right) The value of κ obtained by fitting the eigenvalue distribution to the Gross-Witten form is plotted against T. The dotted line is drawn to guide the eye.

such a way that $\mathrm{tr}\, U$ becomes real positive before taking the expectation value.[7] Using this definition, we have

$$\langle |P| \rangle = \int_{-\pi}^{\pi} d\theta\, \rho(\theta)\, \mathrm{e}^{i\theta} \ . \tag{4.2}$$

Figure 5 (left) shows the result of the eigenvalue distribution $\rho(\theta)$ for $N = 32$. It is clear that the second order phase transition is associated with the emergence of a gap in the eigenvalue distribution.

Thus we find that the "deconfined phase", in which the U(1) symmetry (2.3) is spontaneously broken, is further divided into the gapped phase and the non-uniform phase. In these phases, we may consider the Gross-Witten model [34]

$$Z_{\mathrm{GW}} = \int dU \exp\left\{ \frac{N}{\kappa} \left(\mathrm{tr}\, U + \mathrm{tr}\, U^{\dagger} \right) \right\} \tag{4.3}$$

as a phenomenological model for the holonomy matrix, where the parameter κ should be determined as a function of T. The large-N limit of the eigenvalue distribution for the Gross-Witten model, which we denote as $\rho_{\mathrm{GW}}(\theta)$, is known analytically [34]. For $\kappa < 2$ we have a gapped distribution given by

$$\rho_{\mathrm{GW}}(\theta) = \frac{2}{\pi\kappa} \left(\cos\frac{\theta}{2} \right) \sqrt{\frac{\kappa}{2} - \sin^2\frac{\theta}{2}} \tag{4.4}$$

for $|\theta| \leq 2\sin^{-1}\sqrt{\frac{\kappa}{2}}$ and 0 otherwise. For $\kappa \geq 2$ we have a gapless distribution

$$\rho_{\mathrm{GW}}(\theta) = \frac{1}{2\pi} \left(1 + \frac{2}{\kappa}\cos\theta \right) \ . \tag{4.5}$$

[7] Note that the U(1) transformation (2.3) rotates all the eigenvalues $\mathrm{e}^{i\vartheta_j}$ by some constant angle on the complex plane. If we took the expectation value in eq. (4.1) naively, we would trivially get a uniform distribution.

From these results, one obtains [34]

$$\lim_{N\to\infty}\left\langle \frac{1}{N}\mathrm{tr}\,U\right\rangle_{\mathrm{GW}} = \int_{-\pi}^{\pi} d\theta\,\rho_{\mathrm{GW}}(\theta)\,\mathrm{e}^{i\theta} = \begin{cases} 1-\frac{\kappa}{4} & \text{for } \kappa < 2\,, \\ \frac{1}{\kappa} & \text{for } \kappa \geq 2\,, \end{cases} \tag{4.6}$$

which crosses 1/2 at the critical point $\kappa = 2$. Note that (4.6) and its first derivative with respect to κ is continuous at $\kappa = 2$, but the second derivative has a discontinuity. Thus, the Gross-Witten model undergoes a third order phase transition at $\kappa = 2$.

As we see from figure 5 (left), the distribution $\rho(\theta)$ for $N = 32$ can be nicely fitted to the Gross-Witten form by choosing κ appropriately at each T. The value of κ obtained in this way is plotted against T on the right panel of the same figure. We observe that the first derivative of κ with respect to T is discontinuous at $T = T_{\mathrm{c1}}$. This is reflected in the behavior of the Polyakov line $\langle|P|\rangle$ for $N = 32$ shown in figure 3. Thus it is possible to obtain a second (instead of third) order phase transition between the non-uniform phase and the gapped phase in the present model, despite the fact that the eigenvalue distribution is well described by the Gross-Witten form.

In the "confined phase", the U(1) symmetry (2.3) is unbroken, and therefore the eigenvalue distribution $\rho(\theta)$ is uniform in the large-N limit; hence we call it the uniform phase following the present terminology. However, as we have seen in figure 1, the Polyakov line, which should vanish for a uniform distribution, seems to be of $\mathrm{O}(1/N)$ at low temperature, which actually vanishes slowly with increasing N. On the other hand, the Eguchi-Kawai equivalence, which is a consequence of the unbroken U(1) symmetry, holds with high accuracy at low temperature as we have seen in figure 2. In general the breaking of the Eguchi-Kawai equivalence due to finite N effects is expected to be of $\mathrm{O}(1/N^2)$. Taking advantage of this fact, we identify the uniform phase using the Eguchi-Kawai equivalence instead of identifying it directly using the eigenvalue distribution. In particular, the internal energy normalized by N^2 is constant at low temperature, and the constant value, which gives the ground state energy, is extracted as $\varepsilon_0 = 6.695(5)$ from the $N = 32$ data. With increasing T, one enters the non-uniform phase at some critical point T_{c2}, where the internal energy starts to deviate from this constant value. We fit the results of the internal energy for $N = 32$ shown in figure 4 to the behavior

$$\frac{E}{N^2} - \varepsilon_0 = \begin{cases} 0 & \text{for } T \leq T_{\mathrm{c2}}\,, \\ c(T-T_{\mathrm{c2}})^p & \text{for } T > T_{\mathrm{c2}}\,, \end{cases} \tag{4.7}$$

with three free parameters, which are determined as $c = 413 \pm 310$, $T_{\mathrm{c2}} = 0.8758(9)$ and $p = 2.1(2)$. We redo similar analysis for the "extent of space" $\langle R^2\rangle$. The constant value at low temperature is given by $(r_0)^2 = 2.291(1)$. The deviation $\langle R^2\rangle - (r_0)^2$ can be fitted to the behavior (4.7) with three free parameters, which are determined this time as $c = 39 \pm 36$, $T_{\mathrm{c2}} = 0.8763(4)$ and $p = 1.9(2)$. The results for T_{c2} and p obtained from the two observables are consistent with each other. Moreover they suggest $p = 2$, which implies that the phase transition between the uniform phase and the non-uniform phase is of third order.

Thus we have identified the non-uniform phase between the two critical points $T_{\mathrm{c1}} = 0.905(2)$ and $T_{\mathrm{c2}} = 0.8761(3)$. The range of T is very narrow. This is not so surprising, however, given that the Polyakov line grows very rapidly as we increase the temperature in the critical regime. Note that the Polyakov line should lie within the range $[0, \frac{1}{2}]$ in the non-uniform phase according to the (successful) phenomenological description in terms of the Gross-Witten model.

Let us also comment that the phase structure obtained above is consistent with one of the two scenarios suggested by the Landau-Ginzburg analysis [15]. The other scenario was that the Polyakov line jumps from 0 to 1/2 at some critical temperature, indicating a single first order transition between the uniform phase and the gapped phase. This behavior was observed in the plane-wave matrix model at finite temperature [35, 36]. On the other hand, in ref. [23] we observed no phase transition in the fuzzy sphere background. This might also be the case with the supersymmetric version of the present model [11, 18].

5. Connection to the black-hole/black-string transition

In fact one can show that the model (2.1) describes the high temperature regime of (1+1)d super Yang-Mills theory on a circle. At low-temperature strong-coupling regime, the 2d theory is expected to have a first order phase transition, which corresponds to the black-hole/black-string transition in the dual gravity description. In this section we review this connection, and discuss the implication of our results on it.

Let us consider 2d $\mathrm{U}(N)$ $\mathcal{N}=8$ super Yang-Mills theory,[8] which can be obtained by dimensionally reducing 10d $\mathrm{U}(N)$ $\mathcal{N}=1$ super Yang-Mills theory to 2d. In order to put the theory at finite temperature, we compactify the Euclidean time direction to a circle with the circumference $\tilde{\beta}$, which corresponds to the inverse temperature $\tilde{\beta}=\frac{1}{\tilde{T}}$. Furthermore we compactify the spatial direction to a circle[9] with the circumference $\tilde{L}$. At sufficiently high temperature, the temporal Kaluza-Klein modes decouple, and one obtains our model (2.1) with appropriate identification of parameters. Note in particular that the fermions in the 2d theory decouple due to anti-periodic boundary conditions in the temporal direction, and therefore one ends up with a bosonic theory. The condition for the decoupling of the temporal Kaluza-Klein modes is given by

$$\tilde{T} \gg \left(\frac{\tilde{\lambda}}{\tilde{L}}\right)^{1/3} . \tag{5.1}$$

In this interpretation the t-direction of our 1d model is identified with the spatial direction of the original 2d theory, and the relationship among the parameters is given by

$$\beta = \tilde{L}\,, \quad \lambda = \tilde{\lambda}\tilde{T} . \tag{5.2}$$

Hence the dimensionless effective coupling constant λ_{eff} defined in eq. (2.2) is written as

$$\lambda_{\mathrm{eff}} = \tilde{\lambda}\tilde{T}\tilde{L}^3 . \tag{5.3}$$

[8] We put tildes on all the parameters of the 2d theory to distinguish them from the parameters of our 1d theory.

[9] This model is formally the same as the matrix string theory [37, 38, 5] except that fermions obey anti-periodic boundary conditions in the temporal direction.

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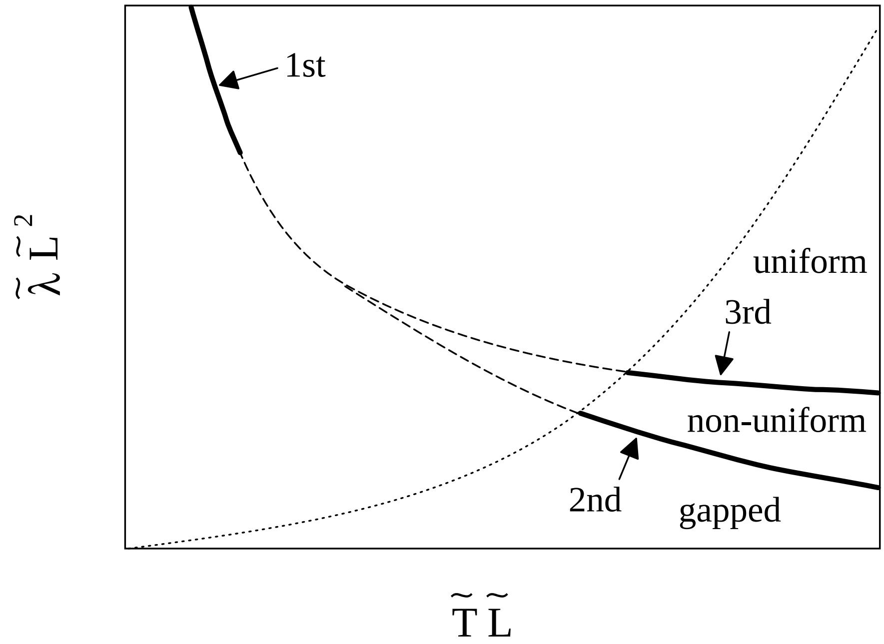

Figure 6: A schematic view of the phase diagram of the 2d super Yang-Mills theory in terms of dimensionless parameters $\tilde{T}\tilde{L}$ and $\tilde{\lambda}\tilde{L}^2$. The region below the dotted line (5.1) can be well described by our 1d model, from which we obtain the critical lines corresponding to (5.4). The upper left corner is conjectured to have a dual gravity description, which predicts the first order phase transition at (5.5). The dashed lines represent our speculation that the non-uniform phase ceases to exist at a tri-critical point for consistency with the prediction from the gauge/gravity duality.

Note also that the Polyakov line (2.4) should be regarded as the Wilson loop winding around a spatial direction in the 2d theory. In ref. [15] the 1d model (2.1) was studied from this point of view and a phase transition was observed around $\lambda_{\text{eff}} \simeq 1.4$. Our results suggest that actually there exist two phase transitions at

$$\lambda_{\text{eff}} \simeq 1.35(1) \quad \text{and} \; \lambda_{\text{eff}} \simeq 1.487(2)\,, \tag{5.4}$$

which are of second order and of third order, respectively.

At large $\tilde{\lambda}$ and small $\tilde{T}$, the 2d theory has a dual gravity description, which predicts a first order phase transition at [15]

$$\tilde{T}\tilde{L} = \frac{2.29}{\sqrt{\tilde{\lambda}\tilde{L}^2}}\;. \tag{5.5}$$

On the gravity side, this corresponds to the black-hole/black-string transition, which is associated with the Gregory-Laflamme instability [39] of the black string winding around the spatial direction. The black-string phase and the black-hole phase can be naturally identified with the uniform phase and the gapped phase on the gauge theory side. A similar correspondence was suggested earlier by ref. [40].

We speculate that the first order phase transition predicted at low temperature actually splits into two continuous transitions as one increases the temperature. In figure 6 we present a schematic view of the phase diagram of the 2d super Yang-Mills theory that emerges from the present work.

6. Summary and discussions

In this paper we have investigated the phase structure of matrix quantum mechanics at finite temperature. We have identified three phases. At high temperature, the high temperature expansion up to the next-leading order gives a precise description of various observables. At low temperature, the internal energy, in particular, does not depend on the temperature as a consequence of the Eguchi-Kawai equivalence. This property enables us to determine the critical point and the order of the transition between the uniform phase and the non-uniform phase.

In the non-uniform phase and the gapped phase, the eigenvalue distribution of the holonomy matrix can be fitted nicely to the Gross-Witten form. While this suggests that we are already seeing the large-N behaviors, it is certainly desirable to confirm the stability of our results against increasing N, which we leave for future investigations.

Our results can be alternatively interpreted as representing the high temperature behavior of 2d $\mathcal{N} = 8$ super Yang-Mills theory. The low temperature behavior of that theory can be predicted by the gauge/gravity correspondence. We speculate that the non-uniform phase identified in the present paper ceases to exist below some temperature in the phase diagram of the 2d super Yang-Mills theory. If so, it is interesting to locate this point explicitly.

The non-uniform phase may exist also in higher dimensional bosonic gauge theories on a finite torus [28, 29]. That will have implications on the phase diagram of super Yang-Mills theories in $D = 3, 4$. The low temperature regime of these theories is discussed in ref. [42] based on the gauge/gravity correspondence.

It would be also interesting to study the supersymmetric version of the present model using the non-lattice simulation proposed recently [43]. In particular, by studying the low temperature regime, we would be able to test the predictions of the gauge/gravity correspondence directly. The phase transitions are expected to disappear [11, 18], and the internal energy is expected to vanish as $T \to 0$ with the power law behavior [13] obtained from the dual black-hole geometry [41]. Indeed the preliminary results for a simplified model with 4 supercharges [43] agree qualitatively with these expectations. Studies of the model with 16 supercharges are in progress [44].

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A. Derivation of a formula for the internal energy

In this section we derive the formula (2.8) relating the internal energy of the present model to the expectation value (2.9), which is directly accessible by Monte Carlo simulation.

Let us first rewrite (2.7) as

$$E = -\frac{1}{Z(\beta)} \lim_{\Delta\beta \to 0} \frac{Z(\beta') - Z(\beta)}{\Delta\beta} , \tag{A.1}$$

where $\beta' = \beta + \Delta\beta$, and represent $Z(\beta')$ for later convenience as

$$Z(\beta') = \int [\mathcal{D}X']_{\beta'} [\mathcal{D}A']_{\beta'} \, \mathrm{e}^{-S'} , \tag{A.2}$$

where S' is obtained from S given in (2.1) by replacing β, t, $A(t)$, $X_i(t)$ with β', t', $A'(t')$, $X'_i(t')$. In order to relate $Z(\beta')$ to $Z(\beta)$, we consider the transformation

$$t' = \frac{\beta'}{\beta} t , \quad A'(t') = \frac{\beta}{\beta'} A(t) , \quad X'_i(t') = \sqrt{\frac{\beta'}{\beta}} X_i(t) . \tag{A.3}$$

The factors in front of the fields are motivated on dimensional grounds, and in particular we have $[\mathcal{D}X']_{\beta'} = [\mathcal{D}X]_\beta$ and $[\mathcal{D}A']_{\beta'} = [\mathcal{D}A]_\beta$. Under this transformation, the kinetic term in S' reduces to that in S, but the interaction term transforms non-trivially as

$$\int_0^{\beta'} dt' \, \mathrm{tr} \left([X'_i(t'), X'_j(t')]^2 \right) = \left(\frac{\beta'}{\beta} \right)^3 \int_0^{\beta} dt \, \mathrm{tr} \left([X_i(t), X_j(t)]^2 \right) . \tag{A.4}$$

This gives us the relation

$$Z(\beta') = Z(\beta) \left\{ 1 - \frac{3}{4} N^2 \Delta\beta \langle F^2 \rangle + \mathrm{O}\Big((\Delta\beta)^2 \Big) \right\} , \tag{A.5}$$

where the operator F^2 is defined by (2.9). Plugging this into (A.1), we get (2.8).

B. Details of Monte Carlo simulation

In this section we present the details of our Monte Carlo simulation. We discretize the Euclidean time direction and obtain the partition function

$$Z_{\mathrm{lat}} = \int [dV][dX_i] \exp(-S_{\mathrm{lat}}) \tag{B.1}$$

$$S_{\mathrm{lat}} = aN \sum_{n=1}^{N_t} \mathrm{tr} \left\{ \frac{1}{2} \left(\frac{V(n) X_i(n+1) V^\dagger(n) - X_i(n)}{a} \right)^2 - \frac{1}{4} [X_i(n), X_j(n)]^2 \right\} , \tag{B.2}$$

where a is the lattice spacing, and the inverse temperature is given by $\beta = aN_t$. The link variables $V(n)$ are $N \times N$ unitary matrices representing the gauge connection between the lattice sites. Due to the periodic boundary conditions, we have $X_i(N_t + 1) = X_i(1)$.

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Although it is possible to simulate the system (B.1) directly, let us simplify it [15] by taking the static diagonal gauge

$$V(1) = V(2) = \cdots = V(N_t) \equiv V = \mathrm{diag}\left(\mathrm{e}^{i\theta_1/N_t}, \mathrm{e}^{i\theta_2/N_t}, \ldots, \mathrm{e}^{i\theta_N/N_t}\right). \tag{B.3}$$

The integration measure for the angular variables $\theta_a \in (-\pi, \pi]$ is given by

$$[d\theta] = \left(\prod_{a=1}^{N} d\theta_a\right) \Delta(\theta)\,, \tag{B.4}$$

$$\Delta(\theta) = \prod_{a<b} \sin^2\left(\frac{\theta_a - \theta_b}{2}\right), \tag{B.5}$$

where $\Delta(\theta)$ is the Vandermonde determinant.

The operators (2.5), (2.9) and (2.10) can be calculated on the lattice by

$$U = V(1)V(2)\cdots V(N_t) = \mathrm{diag}\left(\mathrm{e}^{i\theta_1}, \mathrm{e}^{i\theta_2}, \ldots, \mathrm{e}^{i\theta_N}\right)\,, \tag{B.6}$$

$$F^2 = -\frac{1}{NN_t}\sum_{n=1}^{N_t} \mathrm{tr}\left([X_i(n), X_j(n)]^2\right), \tag{B.7}$$

$$R^2 = \frac{1}{NN_t}\sum_{n=1}^{N_t} \mathrm{tr}\left(X_i(n)^2\right)\,. \tag{B.8}$$

In order to employ the heat-bath algorithm for updating $X_i(n)$, we use the trick used in simulating the bosonic IKKT model [45]. Namely, we introduce $N \times N$ Hermitian matrices $Q_{ij}(n)$ ($i, j = 1, \ldots, 9$; $i < j$) as auxiliary fields, and consider the partition function

$$\tilde{Z}_{\mathrm{lat}} = \int [d\theta][dX][dQ] \exp(-\tilde{S}_{\mathrm{lat}}) \tag{B.9}$$

$$\begin{aligned}\tilde{S}_{\mathrm{lat}} = aN\sum_{n=1}^{N_t} \mathrm{tr}\Bigg[&\frac{1}{2}\left(\frac{VX_i(n+1)V^\dagger - X_i(n)}{a}\right)^2 \\ &+\frac{1}{2}\sum_{i<j}\Big(Q_{ij}(n)^2 - 2Q_{ij}(n)\{X_i(n), X_j(n)\}\Big) \\ &+2\sum_{i<j} X_i(n)^2 X_j(n)^2\Bigg]\,.\end{aligned} \tag{B.10}$$

Integrating out the auxiliary fields $Q_{ij}(n)$, we retrieve the original action (B.2). We update $Q_{ij}(n)$ and $X_i(n)$ for each n in the same way as described in ref. [45].

After updating all the elements of $Q_{ij}(n)$ and $X_i(n)$ for each n, we update the angular variables θ_a using the standard Metropolis algorithm. Typically the acceptance rate is not very high. We therefore repeat the Metropolis procedure sufficiently many times so that most of θ_a get updated. This defines our "one sweep". For $N = 16$ ($N = 32$) we have made 200,000 sweeps (150,000 sweeps) in total, and discarded the first 20,000 sweeps (40,000 sweeps) for thermalization at each temperature. The simulation has been performed on PCs with Pentium 4 (3GHz), and it took a few weeks to get results at each temperature for $N = 32$.

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